Ray, Tom:

Though I cannot expect you to be interested in my little discoveries, I sometimes feel a need to share them with someone, and you have the misfortune to be "it."

While looking to the "asymptotic direction vectors" on the hyperboloid $x^2 + y^2 - z^2 = 1$; *i.e.*, to the 2-vectors that satisfy

$$(\boldsymbol{a}, \mathbb{H}\,\boldsymbol{a}) = 0$$

where

$$\mathbb{H} = \begin{pmatrix} v^2 - 1 & -uv \\ -uv & u^2 - 1 \end{pmatrix}$$

derives from the 2[nd] fundamental form, I was led to the following "Pythagorean identities"

$$\begin{aligned} \Big(v^2 - 1\Big)^2 + \Big(uv + \sqrt{u^2+v^2-1}\,\Big)^2 &= \Big(u + v\sqrt{u^2+v^2-1}\,\Big)^2 \\ \Big(v^2 - 1\Big)^2 + \Big(uv - \sqrt{u^2+v^2-1}\,\Big)^2 &= \Big(u - v\sqrt{u^2+v^2-1}\,\Big)^2 \end{aligned}$$

In the case $\{u, v\} = \{3, 1\}$ the first identity gives $8^2+6^2 = 10^2$; *i.e.*, the $\{3, 4, 5\}$ triangle, but when u and v are assigned other integer values the radicals (in every case, so far as I am aware[1]) mess things up. These identities, in other words, do not—and are not intended—to serve like Euclid's

$$(m^2 - n^2)^2 + (2mn)^2 = (m^2 + n^2)^2$$

Another curiosity: let[2] $P(x, \tau) = 1 + 3x\tau - \tau^3$. Then

$$P(x, -\tau) = \tau^3 \cdot P(x/\tau, -1/\tau)$$

which is curiously reminiscent of Jacobi's identity

$$\vartheta_3(z, \tau) = A \cdot \vartheta_3(z/\tau, -1/\tau) \quad \text{where} \quad A = \sqrt{i/\tau}\, e^{z^2/i\pi\tau}$$

[1] The issue hinges on finding integer solutions of $u^2+v^2-1 = w^2$, a problem that I suspect was solved centuries ago.

[2] I have sketched elsewhere the train of thought that led Ahmed Sebbar from the 2-dimensional theory of unimodular circulant matrices (Pell's problem) to interest in the polynomial $1 - 2xh + h^2$, and in three dimensions to the polynomial $Q(x, h) = 1 + 3xh - h^3$. The polynomial $P(x, h)$ arises from

$$P(x, h) = Q(x/h, h) = -h^3 Q(-x, 1/h)$$

ADDENDUM: Concerning the "problem that must have been solved centuries ago," let

$$w(u,v) = \sqrt{u^2 + v^2 - 1}$$

By quick *Mathematica* search ($1 \leqslant u \leqslant v \leqslant 20$)

$$\begin{aligned} w(4,7) &= 8 \\ w(5,5) &= 7 \\ w(6,17) &= 18 \\ w(7,11) &= 13 \\ w(8,9) &= 12 \\ w(9,19) &= 21 \\ w(10,15) &= 18 \\ w(11,13) &= 17 \\ w(13,19) &= 23 \\ w(14,17) &= 22 \end{aligned}$$

and trivially $w(1,v)$ is an integer for all v (ditto $w(u,1)$, not just in the case $u = 3$ cited). Reversing u and v typically leads to a different Pythagorean triple.